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Lie bi-algebra structures for centrally extended two-dimensional Galilei algebra and their Lie–Poisson counterparts

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Abstract. All bi-algebra structures for centrally extended Galilei algebra are classified. The corresponding Lie–Poisson structures on centrally extended Galilei group are found.

1. Introduction

Recently the problem of deformations of spacetime symmetry groups has attracted much attention [1–7]. One can hope that the deformed groups could provide more general and flexible framework for describing the basic spacetime symmetries. For example, some of them, containing dimensionful deformation parameters could describe the deviation from classical Poincaré symmetry at very high energies providing thereby a natural way of introducing a momentum cut-off for high-energy processes.

In most papers dealing with the subject the relativistic symmetries are considered. However, it seems interesting to understand also the structure of deformed nonrelativistic symmetries. Some preliminary work in this direction has already been done. In particular, in the recent paper [8] all inequivalent bi-algebra structures on two-dimensional Galilei algebra were classified and the corresponding Lie–Poisson structures on the group were found.

From the physical point of view what is really interesting is the central extension of the Galilei algebra. This is because only the genuine projective representations of the Galilei group are relevant in nonrelativistic quantum theory [9]. In the present paper we classify all nonequivalent bi-algebra Lie–Poisson structures for centrally extended two-dimensional Galilei algebra/group. In the two-dimensional case there exists a two-parameter family of central extensions, the parameters being the mass of the particle and the constant force acting on it. We restrict ourselves to the case of free particles, i.e. only the mass parameter is kept nonvanishing.

The paper is organized as follows. First, we find the general form of a 1-cocycle on centrally extended two-dimensional Galilei algebra. Then the action of the most general automorphism transformation on such a 1-cocycle is considered and its orbits are classified, which allows us to find all nonequivalent bi-algebra structures. The corresponding Lie–Poisson structures on Galilei group are then found. The whole procedure follows quite closely the one presented in [10] for $E(2)$ groups and in [8] for two-dimensional Galilei groups. As a result we find 26 nonequivalent bi-algebra structures (some of them still one-parameter families), eight of them being the coboundary ones.

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2. The two-dimensional Galilei group and algebra with central extension and their automorphisms

The two-dimensional Galilei group is a Lie group of transformations of the spacetime with one space dimension. An arbitrary group element g is of the form

$$g = (\tau, v, a) \tag{1}$$

where τ is time translation, a and v are space translation and Galilean boost, respectively. The multiplication law reads

$$g'g = (\tau' + \tau, v' + v, a' + a + \tau v'). \tag{2}$$

The resulting Lie algebra takes the form

$$[K, H] = \mathrm{i}P \qquad [K, P] = 0 \qquad [H, P] = 0. \tag{3}$$

The central extension is obtained by replacing the second commutation rule by

$$[K, P] = \mathrm{i}M \tag{4}$$

where the additional generator M (mass operator) commutes with all other generators,

$$[M, \cdot] = 0. \tag{5}$$

Therefore, we arrive finally at the following algebra

$$[K, H] = \mathrm{i}P \qquad [K, P] = \mathrm{i}M \qquad [H, P] = 0 \qquad [M, \cdot] = 0. \tag{6}$$

Let us define the centrally extended Galilei group by the following global expotential parametrization of group elements

$$\tilde{g} = \mathrm{e}^{\mathrm{i}mM}\mathrm{e}^{-\mathrm{i}\tau H}\mathrm{e}^{\mathrm{i}aP}\mathrm{e}^{\mathrm{i}vK}. \tag{7}$$

Let us write

$$\tilde{g} = (m, \tau, v, a). \tag{8}$$

Then we have the following multiplication law

$$\tilde{g}'\tilde{g} = (m' + m - \tfrac{1}{2}v'^2\tau - av', \tau' + \tau, v' + v, a' + a + \tau v'). \tag{9}$$

The Lie algebra with central extension can be realized in terms of right-invariant fields to be calculated according to the standard rules from the composition law (9)

$$\begin{aligned} X_v^R &= \mathrm{i}\left(\frac{\partial}{\partial v} - a\frac{\partial}{\partial m} + \tau\frac{\partial}{\partial a}\right) \\ X_a^R &= \mathrm{i}\frac{\partial}{\partial a} \\ X_m^R &= \mathrm{i}\frac{\partial}{\partial m} \\ X_\tau^R &= -\mathrm{i}\frac{\partial}{\partial \tau}. \end{aligned} \tag{10}$$

Let us now describe all automorphisms of the algebra (6). The group of automorphisms consists of the following transformations

$$\begin{pmatrix} K \\ H \\ P \\ M \end{pmatrix} \to \begin{pmatrix} K' \\ H' \\ P' \\ M' \end{pmatrix} = \begin{pmatrix} \gamma_3 & \alpha_3 & \beta_3 & \eta_3 \\ 0 & \alpha_1 & \beta_1 & \eta_1 \\ 0 & 0 & \beta_2 & \eta_2 \\ 0 & 0 & 0 & \eta_4 \end{pmatrix} \begin{pmatrix} K \\ H \\ P \\ M \end{pmatrix} \tag{11}$$

where

$$\begin{aligned}\beta_2 &= \gamma_3\alpha_1\\ \eta_2 &= \gamma_3\beta_1\\ \eta_4 &= \gamma_3\beta_2\end{aligned} \tag{12}$$

and, obviously, $\alpha_1 \neq 0$, $\gamma_3 \neq 0$.

3. The bi-algebra structures on two-dimensional centrally extended Galilei algebra

Our aim here is to give a complete classification of Lie bi-algebra structures for the algebra (6) up to automorphisms.

Recall the definition of bi-algebra. It is a pair (L, δ), where L is a Lie algebra while δ is a skewsymmetric cocommutator $\delta : L \to L \otimes L$, i.e.

(i) δ is a 1-cocycle,

$$\delta([X, Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)] \qquad \text{for } X, Y \in L$$

(ii) the dual map $\delta^* : L^* \otimes L^* \to L^*$ defines a Lie bracket on L^*.

We can find all bi-algebra structures on our algebra. The general form of δ obeying (i) is

$$\begin{pmatrix}\delta(H)\\ \delta(P)\\ \delta(K)\\ \delta(M)\end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b & c & d\\ 0 & 0 & 0 & a & 0 & h-b\\ e & 0 & f & g & h & j\\ 0 & 0 & 0 & 0 & 0 & -(a+f)\end{pmatrix}\begin{pmatrix} H\wedge P\\ H\wedge K\\ P\wedge K\\ H\wedge M\\ M\wedge K\\ M\wedge P\end{pmatrix} \tag{13}$$

a, b, c, d, e, f, g, h and j being arbitrary real parameters.

From the condition (ii) we obtain

$$a = b = c = 0$$

or

$$a = e = f = 0 \tag{14}$$

or

$$b = c = f = h = 0.$$

Equations (13) and (14) define all bi-algebra structures on two-dimensional Galilei algebra (6). However, we are interested in classification of nonequivalent bi-algebra structures. To this end we find the transformation rules for the parameters under the automorphisms (11). The counterpart of equation (13) for the transformed generators K', H', P', M' reads

$$\begin{pmatrix}\delta(H')\\ \delta(P')\\ \delta(K')\\ \delta(M')\end{pmatrix} = \begin{pmatrix} \tilde a & 0 & 0 & \tilde b & \tilde c & \tilde d\\ 0 & 0 & 0 & \tilde a & 0 & \tilde h-\tilde b\\ \tilde e & 0 & \tilde f & \tilde g & \tilde h & \tilde j\\ 0 & 0 & 0 & 0 & 0 & -(\tilde a+\tilde f)\end{pmatrix}\begin{pmatrix} H'\wedge P'\\ H'\wedge K'\\ P'\wedge K'\\ H'\wedge M'\\ M'\wedge K'\\ M'\wedge P'\end{pmatrix} \tag{13a}$$

and defines new parameters $\tilde a$, $\tilde b$, $\tilde c$ etc.

By comparing equations (13) and (13a) and using the transformation rule (11) we arrive at the following formulae:

$$
\begin{aligned}
&\tilde{a} = \frac{a}{\beta_2}\\
&\tilde{b} = \frac{b}{\eta_4} + \frac{\alpha_3\alpha_1}{\beta_2\eta_4}c\\
&\tilde{c} = \frac{\alpha_1}{\eta_4\gamma_3}c\\
&\tilde{d} = \frac{\alpha_1}{\beta_2\eta_4}d - 2\frac{\eta_1\alpha_1}{{\beta_2}^3}a + \frac{\alpha_1(\alpha_3\beta_1 - \alpha_1\beta_3)}{{\beta_2}^2\eta_4}c + \frac{\beta_1\eta_2}{{\beta_2}^2\eta_4}a + \frac{\beta_1}{\beta_2\eta_4}h - \frac{\eta_1}{\beta_2\eta_4}f\\
&\tilde{e} = \frac{\gamma_3}{\alpha_1\beta_2}e + \frac{\alpha_3\gamma_3}{{\beta_2}^2}f + \frac{\alpha_3}{\alpha_1\beta_2}a\\
&\tilde{f} = \frac{f}{\beta_2}\\
&\tilde{g} = \frac{\gamma_3}{\alpha_1\eta_4}g - \frac{\beta_1\gamma_3}{\alpha_1{\beta_2}^2}e - \frac{\alpha_3\eta_2}{{\beta_2}^3}f + \frac{\alpha_3}{\alpha_1\eta_4}b + \frac{\alpha_3\gamma_3}{\beta_2\eta_4}h - \frac{\beta_1\alpha_3}{{\beta_2}^2\alpha_1}a + \frac{{\alpha_3}^2}{\beta_2\eta_4}c + \frac{\beta_3}{\alpha_1\eta_4}a\\
&\tilde{j} = \frac{\gamma_3}{\beta_2\eta_4}j - \frac{\eta_1\gamma_3}{{\beta_2}^3}e - \frac{\alpha_3\eta_1}{{\beta_2}^3}f + \frac{\gamma_3\eta_2}{{\beta_2}^2\eta_4}g + \frac{\alpha_3\beta_1}{{\beta_2}^3}h - \frac{\alpha_3\eta_1}{{\beta_2}^3}a + \frac{\alpha_3\eta_2}{{\beta_2}^2\eta_4}b\\
&+\frac{\alpha_3(\alpha_3\beta_1 - \alpha_1\beta_3)}{{\beta_2}^2\eta_4}c + \frac{\alpha_3}{\beta_2\eta_4}d + \frac{\eta_2\beta_3}{{\beta_2}^2\eta_4}a - \frac{\beta_3}{\beta_2\eta_4}b - \frac{\eta_3}{\beta_2\eta_4}a\\
&\tilde{h} = \frac{h}{\eta_4} - \frac{\beta_1}{{\beta_2}^2}f + \frac{\alpha_3}{\eta_4\gamma_3}c\\
&\tilde{h} - \tilde{b} = \frac{h-b}{\eta_4} - \frac{\eta_2}{\beta_2\eta_4}f\\
&\tilde{a} + \tilde{f} = \frac{a+f}{\beta_2}.
\end{aligned}
\tag{15}
$$

We are now in position to classify all orbits of the automorphism group in the space of bi-algebra structures. This is achieved by a straightforward but long and painful analysis of equation (15) (we attempt to nullify as many coefficients as possible on the left-hand side). The resulting complete list of nonequivalent bi-algebra structures is summarized in table 1.

We have checked explictly that all the above bi-algebra structures are consistent and inequivalent. It remains to find coboundary structures (listed also in table 1).

As is well known a cocommutator δ given by

$$\delta(X) = \mathrm{i}[1 \otimes X + X \otimes 1, r] \qquad r \in L \wedge L \qquad X \in L \tag{16}$$

defines a coboundary Lie bi-algebra if and only if r fulfils the modified classical Yang–Baxter equation

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, \xi(r)] = 0 \qquad X \in L \tag{17}$$

where $\xi(r)$ is the Schouten bracket

$$\xi(r) \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$

where

$$
\begin{aligned}
r_{12} &= r^{ij} X_i \otimes X_j \otimes 1\\
r_{13} &= r^{ij} X_i \otimes 1 \otimes X_j\\
r_{23} &= r^{ij} 1 \otimes X_i \otimes X_j.
\end{aligned}
$$

Table 1.

	a	b	c	d	e	f	g	h	j	Remarks
1	0	0	0	0	1	0	0	0	0	
2	0	0	0	0	−1	0	0	0	0	
3	0	0	0	0	0	0	0	1	0	coboundary
4	0	0	0	0	0	0	1	0	0	coboundary
5	0	0	0	1	0	0	0	0	0	coboundary
6	0	0	0	0	0	0	0	0	1	coboundary
7	0	0	0	0	0	0	0	0	−1	coboundary
8	0	0	0	0	0	0	0	1	1	coboundary
9	0	0	0	0	0	0	0	1	−1	coboundary
10	0	0	0	0	1	0	0	1	0	
11	0	0	0	0	−1	0	0	1	0	
12	0	0	0	1	0	0	1	0	0	coboundary
13	0	0	0	0	0	1	0	0	ε	$\varepsilon \in R$
14	0	0	0	0	0	1	1	0	ε	$\varepsilon \in R$
15	0	0	0	1	1	0	0	0	0	
16	0	0	0	1	−1	0	0	0	0	
17	0	1	0	0	0	0	0	0	0	
18	0	1	0	1	0	0	0	0	0	
19	0	ε	0	0	0	0	0	1	0	$\varepsilon \neq 0$
20	0	−1	0	0	0	0	1	1	0	
21	0	ε	1	0	0	0	0	0	0	$\varepsilon \in R$
22	0	ε	1	0	0	0	0	0	1	$\varepsilon \in R$
23	0	ε	1	0	0	0	0	0	−1	$\varepsilon \in R$
24	0	ε	1	0	0	0	1	0	0	$\varepsilon \in R$
25	0	ε	1	0	0	0	−1	0	0	$\varepsilon \in R$
26	1	0	0	0	0	0	0	0	0	

Let us put

$$r = AH \wedge P + CP \wedge K + DH \wedge M + EM \wedge K + FM \wedge P. \tag{18}$$

Equations (18) and (16) now give

$$\begin{aligned} \delta(P) &= -CM \wedge P \\ \delta(H) &= EM \wedge P \\ \delta(K) &= -AH \wedge M - CM \wedge K + DM \wedge P \\ \delta(M) &= 0. \end{aligned} \tag{19}$$

By comparing equations (13) and (19) we get

$$a = b = c = e = f = 0$$
$$A = -g \qquad C = -h \qquad D = j \qquad E = d$$

which serve to identify the coboundary structures in table 1.

4. The Lie–Poisson structures on two-dimensional Galilei group

In this section we find all Lie–Poisson structures on centrally extended two-dimensional Galilei group. Let G be a Lie group, L its Lie algebra and $\{X_i^R\}$ the set of right-invariant fields on G. As is well known

$$\{\Psi, \Phi\} \equiv \eta^{ij}(g) X_i^R \Psi X_j^R \Phi \tag{20}$$

where

$$\eta(g) = \eta^{ij}(g) X_i \otimes X_j, \eta : G \to \Lambda^2 L \tag{21}$$

provides G with a Poisson–Lie group structure if and only if

$$\eta^{il} X_l^R \eta^{jk} + \eta^{kl} X_l^R \eta^{ij} + \eta^{jl} X_l^R \eta^{ki} - c^j_{lp} \eta^{il} \eta^{pk} - c^i_{lp} \eta^{kl} \eta^{pj} - c^k_{lp} \eta^{jl} \eta^{pi} = 0 \tag{22}$$

$$\eta(g'g) = \eta(g') + Adg' \eta(g). \tag{23}$$

In our case let us define

$$\begin{aligned}\eta(m, \tau, a, v) = {} & \lambda(m, \tau, a, v) H \wedge P + \mu(m, \tau, a, v) H \wedge K + \nu(m, \tau, a, v) P \wedge K \\ & + \kappa(m, \tau, a, v) H \wedge M + \rho(m, \tau, a, v) M \wedge K + \pi(m, \tau, a, v) M \wedge P.\end{aligned} \tag{24}$$

Equation (23) gives

$$\begin{aligned}\eta(g'g) = {} & (\lambda' + \lambda - \tau'\mu) H \wedge P + (\nu' + \nu - v'\mu) P \wedge K + (\kappa' + \kappa - v'\lambda + a'\mu) H \wedge M \\ & + (\mu' + \mu) H \wedge K + \left(\rho' + \rho + \frac{v'^2}{2}\mu - v'\nu\right) M \wedge K \\ & + \left(\pi' + \pi - \frac{v'^2}{2}\lambda + \left(v'a' - \frac{v'^2\tau'}{2}\right)\mu + (v'\tau' - a')\nu + v'\kappa - \tau'\rho\right) M \wedge P\end{aligned} \tag{25}$$

where $\lambda' \equiv \lambda(m', \tau', a', v')$ etc.

Consequently we obtain the following set of equations determining λ, μ, ν, κ, ρ and π

$$\begin{aligned}\lambda(m'', \tau'', a'', v'') &= \lambda(m', \tau', a', v') + \lambda(m, \tau, a, v) - \tau'\mu(m, \tau, a, v) \\ \mu(m'', \tau'', a'', v'') &= \mu(m', \tau', a', v') + \mu(m, \tau, a, v) \\ \nu(m'', \tau'', a'', v'') &= \nu(m', \tau', a', v') + \nu(m, \tau, a, v) - v'\mu(m, \tau, a, v) \\ \kappa(m'', \tau'', a'', v'') &= \kappa(m', \tau', a', v') + \kappa(m, \tau, a, v) - v'\lambda(m, \tau, a, v) \\ & \quad + a'\mu(m, \tau, a, v) \\ \rho(m'', \tau'', a'', v'') &= \rho(m', \tau', a', v') + \rho(m, \tau, a, v) + \tfrac{1}{2} v'^2 \mu(m, \tau, a, v) \\ & \quad - v'\nu(m, \tau, a, v) \\ \pi(m'', \tau'', a'', v'') &= \pi(m', \tau', a', v') + \pi(m, \tau, a, v) - \tfrac{1}{2} v'^2 \lambda(m, \tau, a, v) \\ & \quad + v'\kappa(m, \tau, a, v) + (v'a' - \tfrac{1}{2} v'^2 \tau')\mu(m, \tau, a, v) \\ & \quad + (v'\tau' - a')\nu(m, \tau, a, v) - \tau'\rho(m, \tau, a, v).\end{aligned} \tag{26}$$

The strategy to solve equation (26) is to find first the form of η for one-parameter subgroups generated by P, K, H, M and to use again equation (26) as follows. We write the general group element as a product of elements belonging to one-parameter subgroups generated by P, K, H and M, respectively

$$(m, \tau, a, v) = (((m, 0, 0, 0) * (0, \tau, 0, 0)) * ((0, 0, a, 0) * (0, 0, 0, v))) \tag{27}$$

and then we apply equation (26) step by step as indicated on the right-hand side of equation (27) to determine the form of η for general group element.

Equation (26) as specialized for one-parameter subgroups generated by M, H, P and K read

$$\begin{aligned}\lambda(m'', 0, 0, 0) &= \lambda(m', 0, 0, 0) + \lambda(m, 0, 0, 0) \\ \mu(m'', 0, 0, 0) &= \mu(m', 0, 0, 0) + \mu(m, 0, 0, 0)\end{aligned}$$

$$
\begin{aligned}
&\nu(m'', 0, 0, 0) = \nu(m', 0, 0, 0) + \nu(m, 0, 0, 0) \\
&\kappa(m'', 0, 0, 0) = \kappa(m', 0, 0, 0) + \kappa(m, 0, 0, 0) \\
&\rho(m'', 0, 0, 0) = \rho(m', 0, 0, 0) + \rho(m, 0, 0, 0) \\
&\pi(m'', 0, 0, 0) = \pi(m', 0, 0, 0) + \pi(m, 0, 0, 0) \\
&\lambda(0, \tau'', 0, 0) = \lambda(0, \tau', 0, 0) + \lambda(0, \tau, 0, 0) - \tau'\mu(0, \tau, 0, 0) \\
&\mu(0, \tau'', 0, 0) = \mu(0, \tau', 0, 0) + \mu(0, \tau, 0, 0) \\
&\nu(0, \tau'', 0, 0) = \nu(0, \tau', 0, 0) + \nu(0, \tau, 0, 0) \\
&\kappa(0, \tau'', 0, 0) = \kappa(0, \tau', 0, 0) + \kappa(\tau, 0, 0, 0) \\
&\rho(0, \tau'', 0, 0) = \rho(0, \tau', 0, 0) + \rho(0, \tau, 0, 0) \\
&\pi(0, \tau'', 0, 0) = \pi(0, \tau', 0, 0) + \pi(0, \tau, 0, 0) - \tau'\rho(0, \tau, 0, 0) \qquad (28) \\
&\lambda(0, 0, a'', 0) = \lambda(0, 0, a', 0) + \lambda(0, 0, a, 0) \\
&\mu(0, 0, a'', 0) = \mu(0, 0, a', 0) + \mu(0, 0, a, 0) \\
&\nu(0, 0, a'', 0) = \nu(0, 0, a', 0) + \nu(0, 0, a, 0) \\
&\kappa(0, 0, a'', 0) = \kappa(0, 0, a', 0) + \kappa(0, 0, a, 0) + a'\mu(0, 0, a, 0) \\
&\rho(0, 0, a'', 0) = \rho(0, 0, a', 0) + \rho(0, 0, a, 0) \\
&\pi(0, 0, a'', 0) = \pi(0, 0, a', 0) + \pi(0, 0, a, 0) - a'\nu(0, 0, a, 0) \\
&\lambda(0, 0, 0, v'') = \lambda(0, 0, 0, v') + \lambda(0, 0, 0, v) \\
&\mu(0, 0, 0, v'') = \mu(0, 0, 0, v') + \mu(0, 0, 0, v) \\
&\nu(0, 0, 0, v'') = \nu(0, 0, 0, v') + \nu(0, 0, 0, v) - v'\mu(0, 0, 0, v) \\
&\kappa(0, 0, 0, v'') = \kappa(0, 0, 0, v') + \kappa(0, 0, 0, v) - v'\lambda(0, 0, 0, v) \\
&\rho(0, 0, 0, v'') = \rho(0, 0, 0, v') + \rho(0, 0, 0, v) + \tfrac{1}{2}v'^2\mu(0, 0, 0, v) - v'\nu(0, 0, 0, v) \\
&\pi(0, 0, 0, v'') = \pi(0, 0, 0, v') + \pi(0, 0, 0, v) - \tfrac{1}{2}v'^2\lambda(0, 0, 0, v) + v'\kappa(0, 0, 0, v).
\end{aligned}
$$

The corresponding solutions can be readily obtained:

$$
\begin{aligned}
&\lambda(m, 0, 0, 0) = a_1 m \\
&\mu(m, 0, 0, 0) = a_2 m \\
&\nu(m, 0, 0, 0) = a_3 m \\
&\kappa(m, 0, 0, 0) = a_4 m \\
&\rho(m, 0, 0, 0) = a_5 m \\
&\pi(m, 0, 0, 0) = a_6 m \\
&\lambda(0, \tau, 0, 0) = b_1\tau - \tfrac{1}{2}b_2\tau^2 \\
&\mu(0, \tau, 0, 0) = b_2\tau \\
&\nu(0, \tau, 0, 0) = b_3\tau \\
&\kappa(0, \tau, 0, 0) = b_4\tau \\
&\rho(0, \tau, 0, 0) = b_5\tau \\
&\pi(0, \tau, 0, 0) = b_6\tau - \tfrac{1}{2}b_5\tau^2 \qquad (29) \\
&\lambda(0, 0, a, 0) = c_1 a \\
&\mu(0, 0, a, 0) = c_2 a \\
&\nu(0, 0, a, 0) = c_3 a \\
&\kappa(0, 0, a, 0) = c_4 a + \tfrac{1}{2}c_2 a^2
\end{aligned}
$$

$$
\begin{aligned}
&\rho(0,0,a,0) = c_5 a\\
&\pi(0,0,a,0) = c_6 a - \tfrac{1}{2}c_3 a^2\\
&\lambda(0,0,0,v) = d_1 v\\
&\mu(0,0,0,v) = d_2 v\\
&\nu(0,0,0,v) = d_3 v - \tfrac{1}{2}d_2 v^2\\
&\kappa(0,0,0,v) = d_4 v - \tfrac{1}{2}d_1 v^2\\
&\rho(0,0,0,v) = d_5 v + \tfrac{1}{6}d_2 v^3 - \tfrac{1}{2}d_3 v^2\\
&\pi(0,0,0,v) = d_6 v - \tfrac{1}{6}d_1 v^3 + \tfrac{1}{2}d_4 v^2.
\end{aligned}
$$

Now, using equations (29) and (26) we obtain the general form of $\lambda, \mu, \nu, \kappa, \rho$ and π (the resulting expressions were re-inserted back into equation (26) which provided further constraints on the parameters):

$$
\begin{aligned}
&\lambda(m,\tau,a,v) = b_1\tau + d_1 v\\
&\mu(m,\tau,a,v) = 0\\
&\nu(m,\tau,a,v) = d_3 v\\
&\kappa(m,\tau,a,v) = b_4\tau - b_1 a + d_4 v - \tfrac{1}{2}d_1 v^2\\
&\rho(m,\tau,a,v) = b_5\tau + d_5 v - \tfrac{1}{2}d_3 v^2\\
&\pi(m,\tau,a,v) = (b_1-d_3)m + b_6\tau - \tfrac{1}{2}b_5\tau^2 + (b_4+d_5)a + d_6 v - \tfrac{1}{6}d_1 v^3\\
&\qquad + \tfrac{1}{2}d_4 v^2 - d_3 a v - d_5 v\tau + \tfrac{1}{2}d_3\tau v^2.
\end{aligned}
\tag{30}
$$

The general form of η is given by equations (24) and (30). Our next aim is to classify nonequivalent η's. As is well known η defines the bi-algebra structure on L through

$$
\delta(X) = \frac{\mathrm{d}\eta(\mathrm{e}^{\mathrm{i}tx})}{\mathrm{d}t}\Big|_{t=0}. \tag{31}
$$

Simple calculation gives

$$
\begin{aligned}
&\delta(H) = -b_1 H\wedge P - b_4 H\wedge M - b_5 M\wedge K - b_6 M\wedge P\\
&\delta(P) = -b_1 H\wedge M + (b_4+d_5)M\wedge P\\
&\delta(K) = d_1 H\wedge P + d_3 P\wedge K + d_4 H\wedge M + d_5 M\wedge K + d_6 M\wedge P\\
&\delta(M) = (b_1-d_3)M\wedge P.
\end{aligned}
\tag{32}
$$

By comparing equations (13) and (32) we get

$$
\begin{aligned}
&a = -b_1 \qquad e = d_1 \qquad j = d_6\\
&b = -b_4 \qquad f = d_3 \qquad -(a+f) = b_1 - d_3\\
&c = -b_5 \qquad g = d_4 \qquad h - b = b_4 + d_5\\
&d = -b_6 \qquad h = d_5.
\end{aligned}
\tag{33}
$$

Equation (33), together with the results of previous section (table 1) gives us all inequivalent Poisson structures on two-dimensional centrally extended Galilei group. To this end we write out explicitly the general form of the Poisson bracket following from equations (10), (20) and (30)

$$
\begin{aligned}
\{f,g\} = {}& \lambda\left(\frac{\partial f}{\partial\tau}\frac{\partial g}{\partial a} - \frac{\partial f}{\partial a}\frac{\partial g}{\partial\tau}\right) + \kappa\left(\frac{\partial f}{\partial\tau}\frac{\partial g}{\partial m} - \frac{\partial f}{\partial m}\frac{\partial g}{\partial\tau}\right)\\
&+\mu\left(\frac{\partial f}{\partial\tau}\left(\frac{\partial g}{\partial v} - a\frac{\partial g}{\partial m} + \tau\frac{\partial g}{\partial a}\right) - \left(\frac{\partial f}{\partial v} - a\frac{\partial f}{\partial m} + \tau\frac{\partial f}{\partial a}\right)\frac{\partial g}{\partial\tau}\right)
\end{aligned}
$$

Table 2.

	$\{v, a\}$	$\{v, m\}$	$\{\tau, a\}$	$\{\tau, m\}$	$\{a, m\}$	Remarks
1			${\tau_0}^2 v$	$-\frac{1}{2}{\tau_0}^2 v^2$	$-\frac{1}{6}{\tau_0}^2 v^3$	
2			$-{\tau_0}^2 v$	$\frac{1}{2}{\tau_0}^2 v^2$	$\frac{1}{6}{\tau_0}^2 v^3$	
3		${v_0}^2\tau_0 v$			${v_0}^2\tau_0 a$	
4				${\tau_0}^2 v_0 v$	$\frac{1}{2}v_0{\tau_0}^2 v^2$	
5					$-{v_0}^3\tau_0\tau$	
6					${v_0}^2{\tau_0}^2 v$	
7					$-{v_0}^2{\tau_0}^2 v$	
8		${v_0}^2\tau_0 v$			${v_0}^2\tau_0 a + {v_0}^2{\tau_0}^2 v$	
9		${v_0}^2\tau_0 v$			${v_0}^2\tau_0 a - {v_0}^2{\tau_0}^2 v$	
10		${v_0}^2\tau_0 v$	${\tau_0}^2 v$	$-\frac{1}{2}{\tau_0}^2 v^2$	${v_0}^2\tau_0 a - \frac{1}{6}{\tau_0}^2 v^3$	
11		${v_0}^2\tau_0 v$	$-{\tau_0}^2 v$	$\frac{1}{2}{\tau_0}^2 v^2$	${v_0}^2\tau_0 a + \frac{1}{6}{\tau_0}^2 v^3$	
12				${\tau_0}^2 v_0 v$	$\frac{1}{2}v_0{\tau_0}^2 v^2 - {v_0}^3\tau_0\tau$	
13	$\tau_0 v_0 v$	$-\frac{1}{2}\tau_0 v_0 v^2$			$\varepsilon {v_0}^2{\tau_0}^2 v - v_0\tau_0 m$	$\varepsilon \in R$
14	$\tau_0 v_0 v$	$-\frac{1}{2}\tau_0 v_0 v^2$		${\tau_0}^2 v_0 v$	$\varepsilon {v_0}^2{\tau_0}^2 v - v_0\tau_0 m + \frac{1}{2}v_0{\tau_0}^2 v^2$	$\varepsilon \in R$
15			${\tau_0}^2 v$	$-\frac{1}{2}{\tau_0}^2 v^2$	$-{v_0}^3\tau_0\tau - \frac{1}{6}{\tau_0}^2 v^3$	
16			$-{\tau_0}^2 v$	$\frac{1}{2}{\tau_0}^2 v^2$	$-{v_0}^3\tau_0\tau + \frac{1}{6}{\tau_0}^2 v^3$	
17				$-\tau_0{v_0}^2\tau$	$-{v_0}^2\tau_0 a$	
18				$-\tau_0{v_0}^2\tau$	$-\tau_0{v_0}^3\tau - \tau_0{v_0}^2 a$	
19		${v_0}^2\tau_0 v$		$-\varepsilon\tau_0{v_0}^2\tau$	$(1-\varepsilon){v_0}^2\tau_0 a$	$\varepsilon \neq 0$
20		${v_0}^2\tau_0 v$		$\tau_0{v_0}^2\tau + {\tau_0}^2 v_0 v$	$(1+1){v_0}^2\tau_0 a + \frac{1}{2}v_0{\tau_0}^2 v^2$	
21		$-{v_0}^3\tau$		$-\varepsilon\tau_0{v_0}^2\tau$	$-\varepsilon{v_0}^2\tau_0 a - \frac{1}{2}\tau^2{v_0}^3$	$\varepsilon \in R$
22		$-{v_0}^3\tau$		$-\varepsilon\tau_0{v_0}^2\tau$	$-\varepsilon{v_0}^2\tau_0 a - \frac{1}{2}\tau^2{v_0}^3 + {v_0}^2{\tau_0}^2 v$	$\varepsilon \in R$
23		$-{v_0}^3\tau$		$-\varepsilon\tau_0{v_0}^2\tau$	$-\varepsilon{v_0}^2\tau_0 a - \frac{1}{2}\tau^2{v_0}^3 - {v_0}^2{\tau_0}^2 v$	$\varepsilon \in R$
24		$-{v_0}^3\tau$		$-\varepsilon\tau_0{v_0}^2\tau + {\tau_0}^2 v_0 v$	$-\varepsilon{v_0}^2\tau_0 a - \frac{1}{2}\tau^2{v_0}^3 + \frac{1}{2}v_0{\tau_0}^2 v^2$	$\varepsilon \in R$
25		$-{v_0}^3\tau$		$-\varepsilon\tau_0{v_0}^2\tau - {\tau_0}^2 v_0 v$	$-\varepsilon{v_0}^2\tau_0 a - \frac{1}{2}\tau^2{v_0}^3 - \frac{1}{2}v_0{\tau_0}^2 v^2$	$\varepsilon \in R$
26			$-v_0\tau_0\tau$	$\tau_0 v_0 a$	$-v_0\tau_0 m$	

$$
\begin{aligned}
&-\nu\left(\frac{\partial f}{\partial a}\left(\frac{\partial g}{\partial v} - a\frac{\partial g}{\partial m} + \tau\frac{\partial g}{\partial a}\right) - \left(\frac{\partial f}{\partial v} - a\frac{\partial f}{\partial m} + \tau\frac{\partial f}{\partial a}\right)\frac{\partial g}{\partial a}\right)\\
&-\rho\left(\frac{\partial f}{\partial m}\left(\frac{\partial g}{\partial v} - a\frac{\partial g}{\partial m} + \tau\frac{\partial g}{\partial a}\right) - \left(\frac{\partial f}{\partial v} - a\frac{\partial f}{\partial m} + \tau\frac{\partial f}{\partial a}\right)\frac{\partial g}{\partial m}\right)\\
&-\pi\left(\frac{\partial f}{\partial m}\frac{\partial g}{\partial a} - \frac{\partial f}{\partial a}\frac{\partial g}{\partial m}\right).
\end{aligned}
\tag{34}
$$

In particular, the basic Lie–Poisson brackets read

$$
\begin{aligned}
&\{v,\tau\} = -\mu = 0\\
&\{v,a\} = \nu = d_3 v\\
&\{v,m\} = \rho = b_5\tau + d_5 v - \tfrac{1}{2}d_3 v^2\\
&\{\tau,a\} = \lambda + \tau\mu = b_1\tau + d_1 v\\
&\{\tau,m\} = -a\mu + \kappa = b_4\tau - b_1 a + d_4 v - \tfrac{1}{2}d_1 v^2\\
&\{a,m\} = \pi + a\nu + \tau\rho = \tfrac{1}{2}b_5\tau^2 + (b_1 - d_3)m + b_6\tau + (b_4 + d_5)a\\
&+d_6 v - \tfrac{1}{6}d_1 v^3 + \tfrac{1}{2}d_4 v^2.
\end{aligned}
\tag{35}
$$

Obviously, there are further constraints on the parameters following from equation (22) which have not been used yet. Instead of solving it we impose the Jacobi identities on our Poisson brackets (which is equivalent to solving equation (22)). It appears that the additional constaints are, through equation (33), equivalent to those given by equation (14) which provides a further test of the consistency of our results.

Equations (33), (35) and the classification given in table 1 lead us finally to the following classification of nonequivalent Lie–Poisson structures (table 2).

As far as table 2 is concerned the following remark is in order. Up until now we were dealing with dimensionless generators and group parameters. In order to ensure the proper dimensions we replace our generators by dimensionful ones according to the rules

$$H \to \frac{H}{\tau_0} \qquad P \to \frac{P}{v_0\tau_0} \qquad K \to \frac{K}{v_0} \qquad M \to \frac{M}{{v_0}^2\tau_0}$$

where τ_0 and v_0 are arbitrary time and velocity units; the group parameters are redefined appropriately. This redefinition has been already taken into account in table 2.

5. Conclusions

We have classified all inequivalent bi-algebra structures on the centrally extended two-dimensional Galilei algebra and found the corresponding Lie–Poisson structures on the group. The resulting classification appears to be quite rich and contains 26 inequivalent cases, eight of them being the coboundary ones. This is in contrast with semisimple case as well as the case of four-dimensional Poincaré groups in which there are only coboundary structures.

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